

On epicomplete MV -algebras

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Abstract

The aim of the paper is to study epicomplete objects in the category of MV -algebras. A relation between injective MV -algebras and epicomplete MV -algebras is found, an equivalence condition for an MV -algebra to be epicomplete is obtained, and it is shown that the class of divisible MV -algebras and the class of epicomplete MV -algebras are the same. Finally, the concept of an epicompletion for MV -algebras is introduced, and the conditions under which an MV -algebra has an epicompletion are obtained. As a result we show that each MV -algebra has an epicompletion.

AMS Mathematics Subject Classification (2010): 06D35, 06F15, 06F20

Keywords: MV -algebra, Epicomplete MV -algebra, Divisible MV -algebra, Injective MV -algebra, Epicompletion, a -closed MV -algebra.

Acknowledgement: This work was supported by grant VEGA No. 2/0069/16 SAV and GAČR 15-15286S.

1 Introduction

Epicomplete objects are interesting objects in each category. Many researches studied this object in the category of lattice ordered groups (ℓ -group). Pedersen [Ped] defined the concept of an a -epimorphism in this category. It is an ℓ -homomorphism which is also an epimorphism in the category of all torsion free Abelian groups. Anderson and Conrad [AnCo] proved that each epimorphism in the category of Abelian ℓ -groups is an a -epimorphism. They showed that an Abelian ℓ -group G is epicomplete if and only if it is divisible. They also studied epicomplete objects in some subcategories of Abelian ℓ -groups. In particular, they proved that epicomplete objects in the category of Abelian o -groups (linearly ordered groups) with complete o -homomorphisms are the Hahn groups. Darnel [Dar1] continued to study these objects and showed that any completely distributive epicomplete object in the category \mathcal{C} of Abelian ℓ -groups with complete ℓ -homomorphisms is $V(\Gamma, \mathbb{R})$ for some root system Γ , where $V(\Gamma, \mathbb{R})$ is the set of functions $v : \Gamma \rightarrow \mathbb{R}$ whose support satisfies the ascending chain condition with a special order (see [Dar2, Prop 51.2]).

He verified a new subcategory of \mathcal{C} containing completely-distributive Abelian ℓ -groups with complete ℓ -homomorphisms and proved the converse. That is, the only epicomplete objects in this category are of the form $V(\Gamma, \mathbb{R})$. Ton [Ton] studied epicomplete archimedean ℓ -groups and proved that epicomplete objects in this category are ℓ -isomorphic to a semicomplete subdirect sum of real groups. Many other references can be found in [BaHa2, BaHa1]. Recently, Hager [Hag] posed a question on the category of Archimedean ℓ -groups “Does the epicompleteness imply the existence of a compatible reduced f -ring multiplication? ”. His answer to this question was “No” and he tried to find a partial positive answer for it.

MV -algebras were defined by Chang [Cha] as an algebraic counterpart of many-valued reasoning. The principal result of theory of MV -algebras is a representation theorem by Mundici [Mun1] saying that there is a categorical equivalence between the category of MV -algebras and the category of unital Abelian ℓ -groups. Today theory of MV -algebras is very deep and has many interesting connections with other parts of mathematics with many important applications to different areas. For more details on MV -algebras, we recommend the monographs [CDM, Mun2].

In the present paper, epicomplete objects in \mathcal{MV} , the category of MV -algebras, are studied. The concept of an a -extension in \mathcal{MV} is introduced to obtain a condition on minimal prime ideals of an MV -algebra M under which M is epicomplete. Some relations between injective, divisible and epicomplete MV -algebras are found. In the final section, we introduce a completion for an MV -algebra which is epicomplete and has the universal mapping property. We called it the epicompletion and we show that any MV -algebra has an epicompletion.

2 Preliminaries

In the section, we gather some basic notions relevant to MV -algebras and ℓ -groups which will be needed in the next sections. For more details, we recommend to consult the books [AnFe, Dar2] for theory of ℓ -groups and [DiSe, CDM, Mun2] for MV -algebras.

We say that an MV -algebra is an algebra $(M; \oplus, ', 0, 1)$ (and we will write simply $M = (M; \oplus, ', 0, 1)$) of type $(2, 1, 0, 0)$, where $(M; \oplus, 0)$ is a commutative monoid with the neutral element 0 and, for all $x, y \in M$, we have:

- (i) $x'' = x$;
- (ii) $x \oplus 1 = 1$;
- (iii) $x \oplus (x \oplus y')' = y \oplus (y \oplus x')'$.

In any MV -algebra $(M; \oplus, ', 0, 1)$, we can define the following further operations:

$$x \odot y := (x' \oplus y')', \quad x \ominus y := (x' \oplus y)'.$$

In addition, let $x \in M$. For any integer $n \geq 0$, we set

$$0.x = 0, \quad 1.x = x, \quad n.x = (n-1).x \oplus x, \quad n \geq 2,$$

and

$$x^0 = 1, \quad x^1 = x, \quad x^n = x^{n-1} \odot x, \quad n \geq 2.$$

Moreover, the relation $x \leq y \Leftrightarrow x' \oplus y = 1$ is a partial order on M and $(M; \leq)$ is a lattice, where $x \vee y = (x \ominus y) \oplus y$ and $x \wedge y = x \odot (x' \oplus y)$. We use \mathcal{MV} to denote the category of MV -algebras whose objects are MV -algebras and morphisms are MV -homomorphisms. A non-empty subset I of an MV -algebra $(M; \oplus, ', 0, 1)$ is called an *ideal* of M if I is a down set which is closed under \oplus . The set of all ideals of M is denoted by $\mathcal{I}(M)$. For each ideal I of M , the relation θ_I on M defined by $(x, y) \in \theta_I$ if and only if $x \ominus y, y \ominus x \in I$ is a congruence relation on M , and x/I and M/I will denote $\{y \in M \mid (x, y) \in \theta_I\}$ and $\{x/I \mid x \in M\}$, respectively. A *prime* ideal is a proper ideal I of M such that M/I is a linearly ordered MV -algebra, or equivalently, for all $x, y \in M$, $x \ominus y \in I$ or $y \ominus x \in I$. The set of all minimal prime ideals of M is denoted by $\text{Min}(M)$. If M_1 is a subalgebra of an MV -algebra M_2 , we write $M_1 \leq M_2$.

Remark 2.1. Let M_1 be a subalgebra of an MV -algebra M_2 . For any ideal I of M_2 , the set $\bigcup_{x \in M_1} x/I$ is a subalgebra of M_2 containing I which is denoted by $M_1 + I$ for simplicity.

An element a of an MV -algebra $(M; \oplus, ', 0, 1)$ is called a *boolean* element if $a'' := (a')' = a$. The set of all boolean elements of M is denoted by $B(M)$. An ideal I of M is called a *stonean* ideal if there is a subset $S \subseteq B(M)$ such that $I = \downarrow S$, where $\downarrow S = \{x \in M \mid x \leq a \text{ for some } a \in S\}$. An element $x \in M$ is called *archimedean* if there is a integer $n \in \mathbb{N}$ such that $n.x$ is boolean. An MV -algebra M is said to be *hyperarchimedean* if all its elements are archimedean. For more details about hyperarchimedean MV -algebra see [CDM, Chap 6]

A group $(G; +, 0)$ is said to be *partially ordered* if it is equipped with a partial order relation \leq that is compatible with $+$, that is, $a \leq b$ implies $x + a + y \leq x + b + y$ for all $x, y \in G$. An element $x \in G$ is called *positive* if $0 \leq x$. A partially ordered group $(G; +, 0)$ is called a *lattice ordered group* or simply an ℓ -group if G with its partially order relation is a lattice. The *lexicographic product* of two po-groups $(G_1; +, 0)$ and $(G_2; +, 0)$ is the direct product $G_1 \times G_2$ endowed with the lexicographic ordering \leq such that $(g_1, h_1) \leq (g_2, h_2)$ iff $g_1 < g_2$ or $g_1 = g_2$ and $h_1 \leq h_2$ for $(g_1, h_1), (g_2, h_2) \in G_1 \times G_2$. The lexicographic product of po-groups G_1 and G_2 is denoted by $G_1 \times G_2$.

An element u of an ℓ -group $(G; +, 0)$ is called a *strong unit* if, for each $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leq nu$. A couple (G, u) , where G is an ℓ -group and u is a fixed strong unit for G , is said to be a *unital ℓ -group*.

If $(G; +, 0)$ is an Abelian ℓ -group with strong unit u , then the interval $[0, u] := \{g \in G \mid 0 \leq g \leq u\}$ with the operations $x \oplus y := (x + y) \wedge u$ and $x' := u - x$ forms an MV -algebra, which is denoted by $\Gamma(G, u) = ([0, u]; \oplus, ', 0, u)$. Moreover, if $(M; \oplus, ', 0, 1)$ is an MV -algebra, then according to the famous theorem by Mundici, [Mun1], there exists a unique (up to isomorphism) unital Abelian ℓ -group (G, u) with strong u such that $\Gamma(G, u)$ and $(M; \oplus, ', 0, 1)$ are isomorphic (as MV -algebras). Let \mathcal{A} be the category of unital Abelian ℓ -groups whose objects are unital ℓ -groups and morphisms are unital ℓ -group morphisms (i.e. homomorphisms of ℓ -groups preserving fixed strong units). It is important to note that \mathcal{MV} is a variety whereas \mathcal{A} not because it is not closed under infinite products. Then $\Gamma : \mathcal{A} \rightarrow \mathcal{MV}$ is a functor between these categories. Moreover, there is another functor from the category of MV -algebras to \mathcal{A} sending M to a Chang ℓ -group induced by good sequences of the MV -algebra M , which is denoted by $\Xi : \mathcal{MV} \rightarrow \mathcal{A}$. For more details relevant to these functors, please see [CDM, Chaps 2 and 7].

Theorem 2.2. [CDM, Thms 7.1.2, 7.1.7] *The composite functors $\Gamma\Xi$ and $\Xi\Gamma$ are naturally equivalent to the identity functors of \mathcal{MV} and \mathcal{A} , respectively. Therefore, the categories \mathcal{A} and \mathcal{MV} are categorically equivalent.*

Next theorem states that \mathcal{MV} satisfies the amalgamation property.

Theorem 2.3. [Mun2, Thm 2.20] *Given one-to-one homomorphisms $A \xleftarrow{\alpha} Z \xrightarrow{\beta} B$ of MV -algebras, there is an MV -algebra D together with one-to-one homomorphisms $A \xrightarrow{\mu} D \xleftarrow{\nu} B$ such that $\mu \circ \alpha = \nu \circ \beta$.*

An MV -algebra $(M; \oplus, ', 0, 1)$ is called *divisible* if, for all $a \in M$ and all $n \in \mathbb{N}$, there exists $x \in M$ such that

- $n.x = a$.
- $a' \oplus ((n-1).x) = x'$.

Let $(M; \oplus, ', 0, 1)$ be an MV -algebra and (G, u) be the unital Abelian ℓ -group corresponding to M , that is $M = \Gamma(G, u)$. It can be easily seen that M is divisible if and only if, for all $a \in M$ and for all $n \in \mathbb{N}$, there exists $x \in M$ such that the group element nx is defined in M and $nx = a$. Moreover, M is divisible if and only if G is divisible (see [DiSe, Lem. 2.3] or [Glu, Prop 2.13]). It is possible to show that if $nx = a = ny$, then $x = y$ (see [DvRi]). If $(G(M), u)$ is the unital ℓ -group corresponding to an MV -algebra M and $G(M)^d$ is the divisible hull of the ℓ -group $G(M)$, then $G(M)^d$ is an ℓ -group with strong unit u and we use M^d to denote the MV -algebra $\Gamma(G(M)^d, u)$. By [DvRi], M^d is a divisible MV -algebra containing M ; we call M^d the *divisible hull* of M . For more details about divisible MV -algebras we recommend to see [DvRi, DiSe, LaLe].

Definition 2.4. [Glu] An MV -algebra A is *injective* if each MV -homomorphism $h : C \rightarrow A$ from every MV -subalgebra C of an MV -algebra B into A can be extended to an MV -homomorphism from B into A .

Definition 2.5. [DvZa] An ideal I of an MV -algebra M is called a *summand-ideal* if there exists an ideal J of M such that $\langle I \cup J \rangle = M$ and $I \cap J = \{0\}$, where $\langle I \cup J \rangle$ is the ideal of M generated by $I \cup J$. In this case, we write $M = I \boxplus J$. The set of all summand-ideals of M is denoted by $\mathfrak{Sum}(M)$. Evidently, $\{0\}, M \in \mathfrak{Sum}(M)$.

3 Epimorphisms on class of MV -algebras

In this section, epicomplete objects and an epimorphism in the category of MV -algebras are established and their properties are studied. Some relations between epicomplete MV -algebras, a -extensions of MV -algebras and divisible MV -algebras are obtained. We show that any injective MV -algebra is epicomplete. Finally, we prove that an MV -algebra is epicomplete if and only if it is divisible.

Recall that a morphism $f : M_1 \rightarrow M_2$ of \mathcal{MV} is called an *epimorphism* if, for each MV -algebra M_3 and all MV -homomorphisms $\alpha : M_2 \rightarrow M_3$ and $\beta : M_2 \rightarrow M_3$, the condition $\alpha \circ f = \beta \circ f$ implies that $\alpha = \beta$. An object M of \mathcal{MV} is called *epicomplete* if, for each MV -algebra A and for each epimorphism $\alpha : M \rightarrow A$ in \mathcal{MV} , we get α is a surjection. An *epi-extension* for M is an epicomplete object containing M epically. That is, the inclusion map $i : M \rightarrow E$ is an epimorphism and E is an epicomplete MV -algebra.

Definition 3.1. Let M_1 be a subalgebra of an MV -algebra M_2 . Then M_2 is an a -extension of M_1 if the map $f : \mathcal{I}(M_2) \rightarrow \mathcal{I}(M_1)$ defined by $f(J) = J \cap M_1$, $J \in \mathcal{I}(M_2)$, is a lattice isomorphism. An MV -algebra is called a -closed if it has no proper a -extension.

It can be easily seen that M_2 is an a -extension for M_1 if and only if for all $0 < y \in M_2$ there are $n \in \mathbb{N}$ and $0 < x \in M_1$ such that $y < n.x$ and $x < n.y$.

Proposition 3.2. If $f : M_1 \rightarrow M_2$ is an epimorphism, then M_2 is an a -extension for $f(M_1)$.

Proof. Let I and J be two ideals of M_2 such that $I \cap f(M_1) = J \cap f(M_1)$. Then by the Third Isomorphism Theorem [BuSa, Thm 6.18], we get that

$$\frac{M_2}{J} \supseteq \frac{f(M_1) + J}{J} \cong \frac{f(M_1)}{J \cap f(M_1)} = \frac{f(M_1)}{I \cap f(M_1)} \cong \frac{f(M_1) + I}{I} \subseteq \frac{M_2}{I}. \quad (3.1)$$

Let $\alpha_I : \frac{f(M_1)}{I \cap f(M_1)} \rightarrow \frac{M_2}{I}$ and $\alpha_J : \frac{f(M_1)}{J \cap f(M_1)} \rightarrow \frac{M_2}{J}$ be the canonical morphisms induced from (3.1). Then by the amalgamation property (Theorem 2.3), there exist an MV -algebra A and homomorphisms $\beta_I : \frac{M_2}{I} \rightarrow A$ and $\beta_J : \frac{M_2}{J} \rightarrow A$ such that $\beta_I \circ \alpha_I = \beta_J \circ \alpha_J$. Consider the following maps

$$\mu_I : M_2 \xrightarrow{\pi_I} \frac{M_2}{I} \xrightarrow{\beta_I} A, \quad \mu_J : M_2 \xrightarrow{\pi_J} \frac{M_2}{J} \xrightarrow{\beta_J} A,$$

where π_I and π_J are the natural projection homomorphisms. For all $x \in M_1$,

$$\mu_I(f(x)) = \beta_I\left(\frac{f(x)}{I}\right) = \beta_I\left(\alpha_I\left(\frac{f(x)}{I \cap f(M_1)}\right)\right) = \beta_J\left(\alpha_J\left(\frac{f(x)}{J \cap f(M_1)}\right)\right) = \beta_J\left(\frac{f(x)}{J}\right) = \mu_J(f(x)).$$

It follows that $\mu_I \circ f = \mu_J \circ f$ and so by the assumption $\mu_I = \mu_J$, which implies that $I = J$. Therefore, M_2 is an a -extension for $f(M_1)$. \square

The next theorem helps us to prove Corollaries 3.4 and 3.5.

Theorem 3.3. Let M_1 be a subalgebra of an MV -algebra $(M_2; \oplus, ', 0, 1)$ such that M_2 is an a -extension of M_1 and $M_1 + I = M_2$ for all $I \in \text{Min}(M_2)$. Then $M_1 = M_2$.

Proof. Choose $b \in M_2 \setminus M_1$ and set $S := \{x \odot b \mid x \in M_1, x \vee b \in M_1 \text{ and } x \odot b > 0\}$. Clearly, $S \neq \emptyset$ and $0 \notin S$. First we show that S is closed under \wedge . Let $x, y \in M_1$ be such that $x \odot b, y \odot b \in S$. Then $x \vee b, y \vee b \in M_1$ and $x \odot b, y \odot b > 0$. We claim that $(x \wedge y) \odot b > 0$. From [GeIo, Props 1.15, 1.16, 1.21, 1.22] it follows that $(x \wedge y) \odot b = (x \odot b) \wedge (y \odot b)$.

If $(x \wedge y) \odot b = 0$, then

$$x \wedge y \leq b \Rightarrow (x \wedge y) \vee b = b \Rightarrow (x \vee b) \wedge (y \vee b) = b$$

but $(x \vee b) \wedge (y \vee b) \in M_1$ (since $x \vee b, y \vee b \in M_1$), which is a contradiction. So $0 < (x \wedge y) \odot b$. Similarly, we can show that $(x \wedge y) \vee b \in M_1$. Hence $(x \odot b) \wedge (y \odot b) \in S$. It follows that there is a proper lattice filter of M_1 containing S which implies that there exists a maximal lattice filter of M_1 containing S , say \overline{S} , whence $\overline{S} = M_1 \setminus P$ for some minimal prime lattice filter P of M_1 . By [CDM, Cor 6.1.4], P is a minimal prime filter of M_1 , and so there exists $Q \in \text{Min}(M_2)$ such that $P = Q \cap M_1$. By the assumption and by the Third Isomorphism Theorem,

$$\frac{M_1}{Q \cap M_1} \cong \frac{M_1 + Q}{Q} \cong \frac{M_2}{Q}.$$

Then there exists $a \in M_1$ such that $a/Q = b/Q$, so $b \odot a, a \odot b \in Q$. Clearly, $(b \odot a) \vee (a \odot b) \neq 0$ (otherwise, $b = a \in M_1$ which is a contradiction).

(i) If $a \odot b = 0$, then $b \odot a > 0$. Let $0 < b \odot a = t \in Q$. Then there are $n \in \mathbb{N}$ and $z \in M_1$ such that $t < n.z$ and $z < n.t$, so $z, n.z \in Q$ which implies that $n.z \in Q \cap M_1$. From $b \odot a \leq n.z$, we have $b \leq a \oplus n.z$. Clearly, $b < a \oplus n.z$ (since $a \oplus n.z \in M_1$). Thus $(a \oplus n.z) \odot b > 0$ and $(a \oplus n.z) \vee b = a \oplus n.z \in M_1$ and hence by definition

$$(a \oplus n.z) \odot b \in S. \quad (3.2)$$

On the other hand, in view of $\frac{(a \oplus n.z) \odot b}{Q} = (\frac{a}{Q} \oplus \frac{n.z}{Q}) \odot \frac{b}{Q} = \frac{a}{Q} \odot \frac{b}{Q} = \frac{0}{Q}$, we get

$$(a \oplus n.z) \odot b \in Q. \quad (3.3)$$

From relations (3.2) and (3.3) it follows that $(a \oplus n.z) \odot b \in S \cap Q$ which is a contradiction.

(ii) If $b \odot a = 0$, then $b \leq a$ and $a \odot b > 0$, so $a \odot b \in S \cap Q$ (note that $b \vee a = a \in M_1$) which is a contradiction.

(iii) If $b \odot a > 0$ and $a \odot b > 0$, then $a \odot b = t \in Q$, so similarly to (i) there are $n \in \mathbb{N}$ and $z \in M_1$ such that $a \odot b < n.z \in Q \cap M_1$. It follows that $a \odot n.z \leq b$. Since $a \odot n.z \in M_1$, we have $a \odot n.z < b$. Hence, $(a \odot n.z) \odot b = 0$, $\frac{a \odot n.z}{Q \cap M_1} = \frac{a}{Q \cap M_1}$ and $\frac{a \odot n.z}{P} = \frac{b}{P}$. Now, we return to (i) and replace a with $a \odot n.z$. Then we get another contradiction. Therefore, the assumption was incorrect and there is no $b \in M_2 \setminus M_1$. That is, $M_2 = M_1$. \square

Corollary 3.4. *An MV-algebra $(A; \oplus, ', 0, 1)$ is epicomplete if and only if for each epimorphism $f : A \rightarrow B$, we have $\text{Im}(f) + I = B$ for all $I \in \text{Min}(B)$.*

Proof. The proof is straightforward by Proposition 3.2 and Theorem 3.3. \square

Corollary 3.5. *An MV-algebra $(M; \oplus, ', 0, 1)$ is divisible if and only if M/P is divisible for each $P \in \text{Min}(M)$.*

Proof. Consider the divisible hull M^d of the MV-algebra M . For each $n \in \mathbb{N}$ and each $y \in M^d$, there exists $x \in X$ such that $n.x = y$. It entails that $y < (n+1).x$ and $x < (n+1).y$. So, M^d is an a -extension for M . It follows that $\text{Min}(M) = \{P \cap M \mid P \in \text{Min}(M^d)\}$. Moreover, for each $P \in \text{Min}(M^d)$, we have $\frac{M}{P \cap M} = \frac{P+M}{P} \subseteq \frac{M^d}{P}$ and so by the assumption $\frac{P+M}{P}$ is divisible. It follows that $\frac{P+M}{P} = \frac{M^d}{P}$ (since $\frac{M^d}{P}$ is a divisible extension of $\frac{P+M}{P}$), hence $P + M = M^d$. Now, by Theorem 3.3, we conclude that $M = M^d$. Therefore, M is divisible. The proof of the converse is straightforward. \square

In Theorem 3.8, we show a condition under which an MV -algebra is a -closed.

Remark 3.6. If M_2 is an a -extension for MV -algebra M_1 , then for all $I \in I(M_2)$, the MV -algebra $\frac{M_2}{I}$ is an a -extension for the MV -algebra $\frac{M_1+I}{I}$.

Definition 3.7. An ideal I of an MV -algebra $(M; \oplus, ', 0, 1)$ is called an a -ideal if $\frac{M}{I}$ is an a -closed MV -algebra. Clearly, M is an a -closed ideal of M . Moreover, M is a -closed if and only if $\{0\}$ is an a -closed ideal.

Theorem 3.8. *If every minimal prime ideal of an MV -algebra $(M; \oplus, ', 0, 1)$ is a -closed, then M is a -closed.*

Proof. Let A be an a -extension for M . For each $P \in \text{Min}(A)$, we have $\frac{M}{P \cap M} \cong \frac{M+P}{P} \xrightarrow{\subseteq} \frac{A}{P}$. By the above remark, $\frac{A}{P}$ is an a -extension for $\frac{M+P}{P}$. Since $\frac{M+P}{P}$ is a -closed, then $\frac{M+P}{P} = \frac{A}{P}$. Now, from Theorem 3.3, it follows that $M = A$. \square

Clearly, the converse of Theorem 3.8 is true, when M is linearly ordered. Indeed, if M is a chain, $\{0\}$ is the only minimal prime ideal of M and so $M \cong \frac{M}{\{0\}}$ is a -closed. In the following proposition and corollary, we try to find a better condition under which the converse of Theorem 3.8 is true.

Proposition 3.9. *If I is a summand ideal of an a -closed MV -algebra $(M; \oplus, ', 0, 1)$, then $\frac{M}{I}$ is a -closed.*

Proof. Let M be an a -closed MV -algebra and I be a summand ideal of M . By [DvZa, Cor 3.5], there exists $a \in B(M)$ such that $I = \downarrow a$, $I^\perp = \downarrow a'$ and $M = \downarrow a \oplus \downarrow a' := \{x \oplus y \mid x \in \downarrow a, y \in \downarrow a'\}$. Moreover, for each $x \in M$, there are $x_1 \leq a$ and $x_2 \leq a'$ such that $x = x_1 \oplus x_2$ and so $x/I = x_2/I$. Hence for each $x, y \in M$,

$$\begin{aligned} x/I = y/I &\Leftrightarrow x_2/I = y_2/I \Leftrightarrow x_2 \ominus y_2, y_2 \ominus x_2 \in I \\ &\Rightarrow x_2 \ominus y_2 \leq x_2 \in I^\perp, y_2 \ominus x_2 \leq y_2 \in I^\perp \\ &\Rightarrow x_2 \ominus y_2, y_2 \ominus x_2 \in I \cap I^\perp = \{0\} \Rightarrow x_2 = y_2. \end{aligned}$$

That is, $\frac{M}{I} = \{x/I \mid x \in I^\perp\}$. Now, we define the operations \boxplus and $*$ on $\downarrow a'$ by $x \boxplus y = x \oplus y$ and $x^* = t$, where t is the second component of x' in $\downarrow a \oplus \downarrow a'$. It can be easily seen that I^\perp with these operations and 0 and a' as the least and greatest elements, respectively, is an MV -algebra. Moreover, $\frac{M}{I} \cong I^\perp$. Similarly, $\frac{M}{I^\perp}$ is an MV -algebra and $I = \downarrow a \cong \frac{M}{I^\perp}$. Now, let A be an a -extension for the MV -algebra $\frac{M}{I}$. Then

$$\phi : M \xrightarrow{x \mapsto x_1 \oplus x_2} \downarrow a \oplus \downarrow a' \xrightarrow{x \oplus y \mapsto (x/I, y/I^\perp)} \frac{M}{I} \times \frac{M}{I^\perp} \xrightarrow{\subseteq} A \times \frac{M}{I^\perp}.$$

(1) Since $M \cong \frac{M}{I} \times \frac{M}{I^\perp}$, then $\frac{M}{I} \times \frac{M}{I^\perp}$ is a -closed.

(2) $A \times \frac{M}{I^\perp}$ is an a -extension for $\frac{M}{I} \times \frac{M}{I^\perp}$.

It follows that $\frac{M}{I} \times \frac{M}{I^\perp} = A \times \frac{M}{I^\perp}$ and so $A = \frac{M}{I^\perp}$. In a similar way, we can show that M/I is a -closed. \square

Corollary 3.10. *Let $(M; \oplus, ', 0, 1)$ be a closed hyperarchimedean MV -algebra. By [CDM, Thm 6.3.2] every principal ideal of M is a stonean ideal. Hence by [DvZa, Cor 3.5(iii)], we get that every principal ideal of M is a summand ideal of M and so $\frac{M}{I}$ is a -closed for each principal ideal I of M . That is, each principal ideal of M is an a -ideal.*

We note that an MV -algebra M is *simple* if $\mathcal{I}(M) = \{\{0\}, M\}$.

$g = f \circ h$. Then for each $x \in M$, $Id_E \circ f(x) = f(x) = f(h \circ f(x)) = (f \circ h) \circ f(x) = g \circ f(x)$ which implies that $g = Id_E$ (see the latter diagram). Therefore, f is onto. That is, M is epicomplete. \square

It is well known that divisible and complete MV -algebras coincide with injective MV -algebras (see [Lac, Thm 1] and [Glu, Thm 2.14]). So we have the following result.

Corollary 3.13. *Every complete and divisible MV -algebra is epicomplete.*

Let $(M; \oplus, ', 0, 1)$ be an MV -algebra and (G, u) be a unital ℓ -group such that $M = \Gamma(G, u)$. Set $M^d = \Gamma(G^d, u)$, where G^d is the divisible hull of G . Let $i : M \rightarrow M^d$ be the inclusion map. Then i is an epimorphism. Indeed, if A is another MV -algebra and $\alpha, \beta : M^d \rightarrow A$ be MV -homomorphisms such that $\alpha \circ i = \beta \circ i$, then by Theorem 2.2, we have the following homomorphisms in \mathcal{A}

$$\Xi(i) : (G, u) \mapsto (G^d, u), \quad \Xi(\alpha), \Xi(\beta) : \Xi(M^d, u) \mapsto (\Xi(A), v),$$

where v is a strong unit of $\Xi(A)$ such that $\Gamma(\Xi(A), v) = A$. Since Ξ is a functor from \mathcal{MV} to \mathcal{A} , then we have $\Xi(\alpha) \circ \Xi(i) = \Xi(\alpha \circ i) = \Xi(\beta \circ i) = \Xi(\beta) \circ \Xi(i)$. By [AnCo, Sec 2], we know that the inclusion map $\Xi(i) : (G, u) \rightarrow (G^d, u)$ is an epimorphism, so $\Xi(\alpha) = \Xi(\beta)$, which implies that $\alpha = \beta$. Therefore, $i : M \rightarrow M^d$ is an epimorphism in \mathcal{MV} and so we have the following theorem:

Theorem 3.14. *Epicomplete MV -algebras are divisible.*

Theorem 3.15. *Let $(M; \oplus, ', 0, 1)$ be an MV -algebra. Then M is epicomplete if and only if each epimorphism of M into a linearly ordered MV -algebra is onto.*

Proof. Suppose that each epimorphism of M into a linearly ordered MV -algebra H is onto. If $f : M \rightarrow H$ is an epimorphism, then for each $P \in \text{Min}(H)$, the map $M \xrightarrow{f} H \xrightarrow{\pi_P} \frac{H}{P}$ is an epimorphism, where π_P is the natural homomorphism. Since $\frac{H}{P}$ is a linearly ordered MV -algebra, then by the assumption, $\pi_P \circ f$ is onto and so $\frac{f(M)}{P} = \frac{H}{P}$ or equivalently, $\bigcup_{x \in M} f(x)/P = H$. Hence $f(M) + P = H$ for all $P \in \text{Min}(H)$. By Proposition 3.2, we know that H is an a -extension for $f(M)$ and so by Theorem 3.3, $f(M) = H$. Therefore, M is epicomplete. The proof of the other direction is clear. \square

Definition 3.16. Let $(M_1; \oplus, ', 0, 1)$, $(M_2; \oplus, ', 0, 1)$ and $(M_3; \oplus, ', 0, 1)$ be MV -algebras such that $M_1 \leq M_2$ and $M_1 \leq M_3$. An element $b \in M_2$ is *equivalent* to an element $c \in M_3$ if there exists an isomorphism f between $\langle M_1 \cup \{b\} \rangle_{M_2}$ and $\langle M_1 \cup \{c\} \rangle_{M_3}$ such that $f|_{M_1} = Id_{M_1}$, where $\langle M_1 \cup \{b\} \rangle_{M_2}$ is the MV -subalgebra of M_2 generated by $M_1 \cup \{b\}$. The element b is called *algebraic over M_1* if no extension of M_1 contains two elements equivalent to b . Moreover, M_2 is an *algebraic extension* of M_1 if every element of M_2 is algebraic over M_1 .

Proposition 3.17. *Let $(M_1, \oplus, ', 0, 1)$ and $(M_2, \oplus, ', 0, 1)$ be two MV -algebras such that $M_1 \leq M_2$. Then $y \in M_2$ is algebraic over M_1 if and only if the inclusion map $i : M_1 \rightarrow \langle M_1 \cup \{y\} \rangle_{M_2}$ is an epimorphism.*

Proof. Let $y \in M_2$ be algebraic over M_1 . If $i : M_1 \rightarrow \langle M_1 \cup \{y\} \rangle$ is not an epimorphism, then there exist an MV -algebra M_3 and two homomorphisms $\alpha, \beta : \langle M_1 \cup \{y\} \rangle \rightarrow M_3$ such that $\alpha \circ i = \beta \circ i$ and $\alpha \neq \beta$. Then $\alpha(y) \neq \beta(y)$ (otherwise, $\alpha = \beta$). Consider the maps $\lambda, \mu : \langle M_1 \cup \{y\} \rangle_{M_2} \rightarrow M_3 \times \langle M_1 \cup \{y\} \rangle_{M_2}$ defined by $\lambda(x) = (\alpha(x), x)$ and $\mu(x) = (\beta(x), x)$ for all $x \in \langle M_1 \cup \{y\} \rangle_{M_2}$. Clearly, λ and μ are one-to-one homomorphisms. We have $\lambda(M_1) = \{(\alpha(x), x) \mid x \in M_1\} = \mu(M_1)$ and $M_1 \cong \lambda(M_1) \cong \mu(M_1)$. Set $M = M_3 \times \langle M_1 \cup \{y\} \rangle_{M_2}$. We identify M_1 with its image in M under λ . Then M is an extension for M_1 . Since $\lambda : \langle M_1 \cup \{y\} \rangle_{M_2} \rightarrow \langle M_1 \cup \{(\alpha(y), y)\} \rangle_M$ and $\mu : \langle M_1 \cup \{y\} \rangle_{M_2} \rightarrow \langle M_1 \cup \{(\beta(y), y)\} \rangle_M$ are isomorphisms, then y is equivalent to $(\alpha(y), y)$ and $(\beta(y), y)$, which is a contradiction. Therefore, $i : M_1 \rightarrow \langle M_1 \cup \{y\} \rangle$ is an epimorphism. The proof of the converse is clear. \square

Corollary 3.13 showed that complete and divisible MV -algebras are epicomplete. In the sequel, we will use the same argument as in the proof of [AnCo, Thm 2.1] with a little modification to show that every divisible MV -algebra is epicomplete.

Theorem 3.18. *Let $(M; \oplus, ', 0, 1)$ be an MV-algebra.*

- (i) *If $(L; \oplus, ', 0, 1)$ is a linearly ordered MV-algebra and $f : M \rightarrow L$ is an epimorphism, then $L \subseteq f(M)^d$.*
- (ii) *If M is divisible, then it is epicomplete.*

Proof. (i) Let (G, u) and (H, v) be the unital ℓ -groups such that $\Gamma(G, u) = M$ and $\Gamma(H, v) = L$. By Theorem 2.2, $\Xi(f) : G \rightarrow H$ is a unital ℓ -group homomorphism. Let $K = B/\text{Im}(\Xi(f))$ be the torsion subgroup of $H/\text{Im}(\Xi(f))$ (clearly B is a subgroup of G and $x \in B \Leftrightarrow nx \in \text{Im}(\Xi(f))$ for some $n \in \mathbb{N}$). Since $H/B \cong \frac{H/\text{Im}(\Xi(f))}{B/\text{Im}(\Xi(f))}$ is torsion free, by [AnFe, Prop 1.1.7], H/B admits a linearly ordered group structure. By [GlHo, Exm 3], $(H/B) \xrightarrow{\gamma} H$ is an ℓ -group. Since v is a strong unit of an ℓ -group H , for each $(x + B, y) \in (H/B) \xrightarrow{\gamma} H$, there exists $n \in \mathbb{N}$ such that $x, y < nv$ and so $(x + B, y) < n(v + B, v)$. It follows that $w := (v + B, v)$ is a strong unit for the ℓ -group $(H/B) \xrightarrow{\gamma} H$. Let $\alpha, \beta : L \rightarrow \Gamma((H/B) \xrightarrow{\gamma} H, w)$ be defined by $\alpha(x) = (B, x)$ and $\beta(x) = (x + B, x)$ for all $x \in L$ (both of them are MV-homomorphisms). We have $\alpha \circ f(x) = (B, f(x)) = \beta \circ f(x)$ for all $x \in L$. Since f is an epimorphism, then $\alpha = \beta$, hence for all $x \in L = \Gamma(H, v)$, we have $x \in B$ and so there is $n \in \mathbb{N}$ such that $nx \in \text{Im}(\Xi(f))$. That is, x belongs to the divisible hull of $\text{Im}(\Xi(f))$. Since $x \leq v$, then $x \in \Gamma((\text{Im}(\Xi(f)))^d, v) = f(M)^d$. Therefore, $L \subseteq f(M)^d$.

(ii) Let M be divisible. We use Theorem 3.15 to show that M is epicomplete. Let $(A; \oplus, ', 0, 1)$ be a linearly ordered MV-algebra and $f : M \rightarrow A$ be an epimorphism into A . By (i), $f(M) \subseteq L \subseteq (f(M))^d$. Clearly, $f(M)$ is divisible, so $f(M) = L = (f(M))^d$. It follows from Theorem 3.15 that M is epicomplete. \square

Concerning the proof of (i) in the latter theorem, we note that since $f : M \rightarrow f(M)$ is onto, by [Mun1, Lem. 7.2.1], $\Xi(f) : \Xi(M) \rightarrow \Xi(f(M))$ is onto ($\Xi(M) = G$), so $\text{Im}(\Xi(f)) = \Xi(f(M))$. It follows that $f(M)^d = \Gamma((\Xi(f(M)))^d, v) = \Gamma((\text{Im}(\Xi(f)))^d, v)$.

4 Epicompletion of MV-algebras

The main purpose of the section is to introduce an epicompletion for an MV-algebra and to discuss about conditions when an MV-algebra has an epicompletion. First we introduce an epicompletion in \mathcal{MV} . An epicompletion for an MV-algebra A is an MV-algebra M epically containing A with the universal property. Then we use some results of the second section and prove that any MV-algebra has an epicompletion. Indeed, the epicompletion of A is A^d .

Definition 4.1. Let $(A; \oplus, ', 0, 1)$ be an MV-algebra.

- (i) A pair (\overline{A}, α) , where \overline{A} is an MV-algebra and $\alpha : A \rightarrow \overline{A}$ is a one-to-one epimorphism (epiembedding for short), is called an *e-extension* for A .
- (ii) An *e-extension* (E, α) for A is called an *epicompletion* for A if, for each epimorphism $f : A \rightarrow B$, there is an *e-extension* (\overline{B}, β) for B and a surjective homomorphism $\mathbf{f} : E \rightarrow \overline{B}$ such that $\beta \circ f = \mathbf{f} \circ \alpha$, or equivalently, the diagram given by Figure 2 commutes.

Proposition 4.2. *Let $(A; \oplus, ', 0, 1)$ be an MV-algebra.*

- (i) *Each epicompletion of A is epicomplete.*
- (ii) *If A has an epicompletion, then it is unique up to isomorphism.*

Proof. (i) Let (\overline{A}, α) be an epicompletion for A and $f : \overline{A} \rightarrow B$ be an epimorphism. Then $f \circ \alpha : A \rightarrow B$ is an epimorphism and so there exists an *e-extension* (\overline{B}, β) for B and an onto morphism $h : \overline{A} \rightarrow \overline{B}$ such that the diagram in Figure 3 commutes.

$$\begin{array}{ccc}
A & \xrightarrow{f \circ \alpha} & B \\
\alpha \downarrow & & \downarrow \beta \\
E & \xrightarrow{f} & \overline{B}
\end{array}$$

Figure 2:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\alpha \downarrow & & \downarrow \beta \\
\overline{A} & \xrightarrow{h} & \overline{B}
\end{array}$$

Figure 3:

From $h \circ \alpha = \beta \circ f \circ \alpha$ it follows that $h = \beta \circ f$, whence $\beta \circ f$ is onto. Hence $\beta(f(\overline{A})) = \overline{B}$. Also, $\beta(f(\overline{A})) \subseteq \beta(B) \subseteq \overline{B}$, so $\beta(f(\overline{A})) = \beta(\overline{B})$. Since β is one-to-one, f must be onto. Therefore, \overline{A} is epicomplete.

(ii) Let (\overline{A}, α) and (A', β) be two epicompletions for A . Since $\alpha : A \rightarrow \overline{A}$ is an epiembedding, then by Proposition 3.2, \overline{A} is an a -extension for $f(A) \cong A$. By (i) and Theorem 3.14, \overline{A} is divisible. Consider the functor Ξ from Theorem 2.2. Let $(G(A), u)$ and $(G(\overline{A}), v)$ be the unital ℓ -groups induced from MV -algebras A and \overline{A} , respectively. Since there is an epiembedding $A \hookrightarrow \overline{A}$, we have $u = v$. We have an embedding $\Xi(\alpha) : G(A) \rightarrow G(\overline{A})$, and $G(\overline{A})$ is divisible (see [DvRi]). Also, $G(\overline{A})$ is an a -extension for the ℓ -group $G(A)$ (since there is a one-to-one correspondence between the lattice of ideals of A and the lattice of convex ℓ -subgroups of $G(A)$, see [CiTo, Thm 1.2]), so by [Gri, Chap 1, Thm 20], $G(\overline{A}) \cong (G(A))^d$. In a similar way, $G(A') \cong (G(A))^d$. Therefore, $G(\overline{A}) \cong G(A')$. We note that the final isomorphism is an extension for the identity map on G , so it preserves the strong units of $G(\overline{A})$ and $G(A')$, which is a strong unit of $G(A)$, too. Using the functor Γ , it follows that $\overline{A} \cong A'$. \square

Let $(A; \oplus, ', 0, 1)$ be an MV -algebra. By the last proposition if A has an epicompletion, then it is unique up to isomorphic image; this epicompletion is denoted by (A^e, α) .

Corollary 4.3. *Let (A^e, α) be an epicompletion for an MV -algebra $(A; \oplus, ', 0, 1)$. Then the epicompletion of A^e is equal to A^e .*

Proof. It follows from Proposition 4.2(i). \square

Now, we try to answer to a question “whether does an MV -algebra have an epicompletion”. First we simply use Theorem 3.18(i) to show that each linearly ordered MV -algebra has an epicompletion. Then we prove it for any MV -algebra. For this purpose we try to extend the result of [Ped, Cor 1]. We show that each ℓ -group has an epicompletion. Then we use this result and we show that any MV -algebra has an epicompletion.

Proposition 4.4. *Let $(A; \oplus, ', 0, 1)$ be a linearly ordered MV -algebra. Then A has an epicompletion.*

Proof. Let $f : A \rightarrow B$ be an epimorphism. Since A is a chain, $f(A)$ is also a chain and so, $\mathcal{I}(f(M))$ is a chain. It follows from Proposition 3.2 that $\mathcal{I}(B)$ is a chain and so B is a linearly ordered MV -algebra. Hence by Theorem 3.18(i), $B \subseteq (f(A))^d$ (thus $B^d = (f(A))^d$). By [Ped, Prop 5] and Theorem 2.2, there is a homomorphism $g : A^d \rightarrow B^d$ such that the diagram in Figure 4 commutes. $g(A^d) \subseteq B^d$ is a divisible MV -algebra containing B , so g is onto. Therefore, A^d is an epicompletion for the MV -algebra A . \square

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\subseteq \downarrow & & \downarrow \subseteq \\
A^d & \xrightarrow{g} & B^d
\end{array}$$

Figure 4:

Remark 4.5. Let G and H be two ℓ -groups and $f : G \rightarrow H$ be an epimorphism. Let G^d and H^d be the divisible hull of G and H , respectively. By [AnCo, p. 230], there is a unique extension of f to an epimorphism $\bar{f} : G^d \rightarrow H^d$. Clearly, if G and H are unital ℓ -groups and f is a unital ℓ -group morphism, then so is \bar{f} (for more details see [AnCo] the paragraph after Theorem 2.1 and [Ped, Prop 5]). We know that the inclusion maps $i : G \rightarrow G^d$ and $j : H \rightarrow H^d$ are epimorphisms (by the corollary of [AnCo, Thm 2.1]), hence we have the following commutative diagram (Figure 5).

$$\begin{array}{ccc}
G & \xrightarrow{i} & G^d \\
f \downarrow & & \downarrow \bar{f} \\
H & \xrightarrow{j} & H^d
\end{array}$$

Figure 5:

Since $\bar{f} : G^d \rightarrow H^d$ is an epimorphism and G^d is epicomplete (by [AnCo, Thm 2.1]), then \bar{f} is onto and so G^d is an epicompletion for G .

Theorem 4.6. *Any MV-algebra has an epicompletion.*

Proof. Let $(A; \oplus, ', 0, 1)$ be an MV-algebra and $f : A \rightarrow B$ be an epimorphism. Then $\Xi(f) : \Xi(A) \rightarrow \Xi(B)$ is a homomorphism of unital ℓ -groups. By Remark 4.5, we have the commutative diagram in Figure 6, where $\overline{\Xi(f)}$ is the unique extension of $\Xi(f)$. Applying the functor Γ to diagram in Figure 6, we get

$$\begin{array}{ccc}
\Xi(A) & \xrightarrow{\Xi(f)} & \Xi(B) \\
\subseteq \downarrow & & \downarrow \subseteq \\
(\Xi(A))^d & \xrightarrow{\overline{\Xi(f)}} & (\Xi(B))^d
\end{array}$$

Figure 6:

the commutative diagram in Figure 7 on \mathcal{MV} . Set $F := \Gamma(\overline{\Xi(f)})$. We claim that $F : A^d \rightarrow B^d$ is an epimorphism. Let $\alpha, \beta : B^d \rightarrow C$ be two homomorphisms of MV-algebras such that $\alpha \circ F = \beta \circ F$. Then clearly, $\alpha|_B \circ F = \beta|_B \circ F$, so by the assumption $\alpha|_B = \beta|_B$, which implies that $\Xi(\alpha|_B) = \Xi(\beta|_B)$. Thus by [AnCo], $\overline{\Xi(\alpha|_B)} = \overline{\Xi(\beta|_B)}$, where $\overline{\Xi(\alpha|_B)}, \overline{\Xi(\beta|_B)} : (\Xi(B))^d \rightarrow (\Xi(C))^d$ are the unique extensions of $\Xi(\alpha|_B)$ and $\Xi(\beta|_B)$, respectively. It can be easily seen that the diagrams in Figure 8 are commutative. So by the uniqueness of the extension of $\Xi(\alpha|_B) : \Xi(B) \rightarrow \Xi(C)$ to a map $(\Xi(B))^d \rightarrow (\Xi(C))^d$, we get that $\Xi(\alpha) = \overline{\Xi(\alpha|_B)}$. In a similar way, $\Xi(\beta) = \overline{\Xi(\beta|_B)}$ and so $\Xi(\alpha) = \Xi(\beta)$. It follows that $\alpha = \Gamma(\Xi(\alpha)) = \Gamma(\Xi(\beta)) = \beta$. Therefore, F is an epimorphism. Since A^d is divisible, by Corollary 3.13, it is epicomplete and so F is onto. That is, A^d is an epicompletion for A . Therefore, any MV-algebra has an epicompletion. \square

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\subseteq \downarrow & & \downarrow \subseteq \\
A^d & \xrightarrow{\Gamma(\Xi(f))} & B^d
\end{array}$$

Figure 7:

$$\begin{array}{ccc}
\Xi(B) & \xrightarrow{\Xi(\alpha|_B)} & \Xi(C) \\
\subseteq \downarrow & & \downarrow \subseteq \\
(\Xi(B))^d & \xrightarrow{\overline{\Xi(\alpha|_B)}} & (\Xi(C))^d
\end{array}
\qquad
\begin{array}{ccc}
\Xi(B) & \xrightarrow{\Xi(\alpha|_B)} & \Xi(C) \\
\subseteq \downarrow & & \downarrow \subseteq \\
(\Xi(B))^d & \xrightarrow{\Xi(\alpha)} & (\Xi(C))^d
\end{array}$$

Figure 8:

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